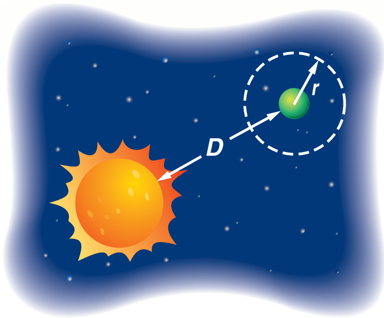
7-6 Rational Exponents

Main Ideas

- Write expressions with rational exponents in radical form and vice versa.
- Simplify expressions in exponential or radical form.

GET READY for the Lesson

Astronomers refer to the space around a planet where the planet's gravity is stronger than the Sun's as the *sphere of influence* of the planet. The radius r of the sphere of influence is given by the formula $r = D\left(\frac{M_p}{M_S}\right)^{\frac{2}{5}}$, where M_p is the mass of the planet, M_S is the mass of the Sun, and D is the distance between the planet and the Sun.



Rational Exponents and Radicals You know that squaring a number and taking the square root of a number are inverse operations. But how would you evaluate an expression that contains a fractional exponent such as the one above? You can investigate such an expression by assuming that fractional exponents behave as integral exponents.

$$\left(b^{\frac{1}{2}}\right)^2 = b^{\frac{1}{2}} \cdot b^{\frac{1}{2}} \quad \text{Write the square as multiplication.}$$

$$= b^{\frac{1}{2}+\frac{1}{2}} \quad \text{Add the exponents.}$$

$$= b^1 \text{ or } b \quad \text{Simplify.}$$

Thus, $b^{\frac{1}{2}}$ is a number whose square equals b. So $b^{\frac{1}{2}} = \sqrt{b}$.

KEY CONCEPT — $b^{\frac{1}{n}}$

Words For any real number b and for any positive integer n, $b^{\frac{1}{n}} = \sqrt[n]{b}$, except when $b < 0$ and n is even.

Example $8^{\frac{1}{3}} = \sqrt[3]{8}$ or 2

EXAMPLE 1 Radical and Exponential Forms

a. Write $a^{\frac{1}{4}}$ in radical form.

$a^{\frac{1}{4}} = \sqrt[4]{a}$ Definition of $b^{\frac{1}{n}}$

b. Write $\sqrt[3]{y}$ in exponential form.

$\sqrt[3]{y} = y^{\frac{1}{3}}$ Definition of $b^{\frac{1}{n}}$

CHECK Your Progress

1A. $x^{\frac{1}{5}}$

1B. $\sqrt[8]{c}$

Study Tip

Negative Base

Suppose the base of a monomial is negative, such as $(-9)^2$ or $(-9)^3$. The expression is undefined if the exponent is even because there is no number that, when multiplied an even number of times, results in a negative number. However, the expression is defined for an odd exponent.

EXAMPLE 2 Evaluate Expressions with Rational Exponents

Evaluate each expression.

a. $16^{-\frac{1}{4}}$

Method 1

$16^{-\frac{1}{4}} = \frac{1}{16^{\frac{1}{4}}}$ — $b^{-n} = \frac{1}{b^n}$

$= \frac{1}{\sqrt[4]{16}}$ — $16^{\frac{1}{4}} = \sqrt[4]{16}$

$= \frac{1}{\sqrt[4]{2^4}}$ — $16 = 2^4$

$= \frac{1}{2}$ — Simplify.

Method 2

$16^{-\frac{1}{4}} = (2^4)^{-\frac{1}{4}}$ — $16 = 2^4$

$= 2^{4\left(-\frac{1}{4}\right)}$ — Power of a Power

$= 2^{-1}$ — Multiply exponents.

$= \frac{1}{2}$ — $2^{-1} = \frac{1}{2^1}$

b. $243^{\frac{3}{5}}$

Method 1

$243^{\frac{3}{5}} = 243^{3\left(\frac{1}{5}\right)}$ — Factor.

$= (243^3)^{\frac{1}{5}}$ — Power of a Power

$= \sqrt[5]{243^3}$ — $b^{\frac{1}{5}} = \sqrt[5]{b}$

$= \sqrt[5]{(3^5)^3}$ — $243 = 3^5$

$= \sqrt[5]{3^5 \cdot 3^5 \cdot 3^5}$ — Expand the cube.

$= 3 \cdot 3 \cdot 3$ or 27 — Find the fifth root.

Method 2

$243^{\frac{3}{5}} = (3^5)^{\frac{3}{5}}$ — $243 = 3^5$

$= 3^{5\left(\frac{3}{5}\right)}$ — Power of a Power

$= 3^3$ — Multiply exponents.

$= 27$ — $3^3 = 3 \cdot 3 \cdot 3$

CHECK Your Progress

2A. $27^{-\frac{1}{3}}$ **2B.** $64^{\frac{2}{3}}$

In Example 2b, Method 1 uses a combination of the definition of $b^{\frac{1}{n}}$ and the properties of powers. This example suggests the following general definition of rational exponents.

KEY CONCEPT *Rational Exponents*

Words For any nonzero real number b, and any integers m and n, with $n > 1$, $b^{\frac{m}{n}} = \sqrt[n]{b^m} = \left(\sqrt[n]{b}\right)^m$, except when $b < 0$ and n is even.

Example $8^{\frac{2}{3}} = \sqrt[3]{8^2} = \left(\sqrt[3]{8}\right)^2$, or 4

In general, we define $b^{\frac{m}{n}}$ as $\left(b^{\frac{1}{n}}\right)^m$ or $(b^m)^{\frac{1}{n}}$. Now apply the definition of $b^{\frac{1}{n}}$ to $\left(b^{\frac{1}{n}}\right)^m$ and $(b^m)^{\frac{1}{n}}$.

$$\left(b^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{b}\right)^m \qquad (b^m)^{\frac{1}{n}} = \sqrt[n]{b^m}$$

Real-World EXAMPLE

3 **WEIGHTLIFTING** **The formula $M = 512 - 146{,}230B^{-\frac{8}{5}}$ can be used to estimate the maximum total mass that a weightlifter of mass B kilograms can lift using the snatch and the clean and jerk.**

a. **According to the formula, what is the maximum amount that 2004 Olympic champion Hossein Reza Zadeh of Iran can lift if he weighs 163 kilograms?**

$M = 512 - 146{,}230B^{-\frac{8}{5}}$ — Original formula

$= 512 - 146{,}230(163)^{-\frac{8}{5}}$ or about 470 — $B = 163$

The formula predicts that he can lift at most 470 kilograms.

b. **Hossein Reza Zadeh's winning total in the 2004 Olympics was 472.50 kg. Compare this to the value predicted by the formula.**

The formula prediction is close to the actual weight, but slightly lower.

CHECK Your Progress

3. The radius r of a sphere with volume V is given by $r = \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}}$. Find the radius of a ball whose volume is 77 cm^3.

Real-World Link

With origins in both the ancient Egyptian and Greek societies, weightlifting was among the sports on the program of the first Modern Olympic Games, in 1896, in Athens, Greece.

Source: International Weightlifting Association

Simplify Expressions All of the properties of powers you learned in Lesson 6-1 apply to rational exponents. When simplifying expressions containing rational exponents, leave the exponent in rational form rather than writing the expression as a radical. Write the expression with all positive exponents. Also, any exponents in the denominator of a fraction must be positive *integers*. So, it may be necessary to rationalize a denominator.

EXAMPLE Simplify Expressions with Rational Exponents

4 **Simplify each expression.**

a. $x^{\frac{1}{5}} \cdot x^{\frac{7}{5}}$

$x^{\frac{1}{5}} \cdot x^{\frac{7}{5}} = x^{\frac{1}{5} + \frac{7}{5}}$ — Multiply powers.

$= x^{\frac{8}{5}}$ — Add exponents.

b. $y^{-\frac{3}{4}}$

$y^{-\frac{3}{4}} = \frac{1}{y^{\frac{3}{4}}}$ — $b^{-n} = \frac{1}{b^n}$

$= \frac{1}{y^{\frac{3}{4}}} \cdot \frac{y^{\frac{1}{4}}}{y^{\frac{1}{4}}}$ — Why use $\frac{y^{\frac{1}{4}}}{y^{\frac{1}{4}}}$?

$= \frac{y^{\frac{1}{4}}}{y^{\frac{4}{4}}}$ — $y^{\frac{3}{4}} \cdot y^{\frac{1}{4}} = y^{\frac{3}{4} + \frac{1}{4}}$

$= \frac{y^{\frac{1}{4}}}{y}$ — $y^{\frac{4}{4}} = y^1$ or y

CHECK Your Progress

4A. $a^{\frac{1}{4}} \cdot a^{\frac{9}{4}}$

4B. $r^{-\frac{4}{5}}$

Study Tip

Indices

When simplifying a radical expression, always use the smallest index possible.

EXAMPLE Simplify Radical Expressions

5 Simplify each expression.

a. $\dfrac{\sqrt[8]{81}}{\sqrt[6]{3}}$

$\dfrac{\sqrt[8]{81}}{\sqrt[6]{3}} = \dfrac{81^{\frac{1}{8}}}{3^{\frac{1}{6}}}$ Rational exponents

$= \dfrac{(3^4)^{\frac{1}{8}}}{3^{\frac{1}{6}}}$ $81 = 3^4$

$= \dfrac{3^{\frac{1}{2}}}{3^{\frac{1}{6}}}$ Power of a Power

$= 3^{\frac{1}{2} - \frac{1}{6}}$ Quotient of Powers

$= 3^{\frac{1}{3}}$ Simplify.

$= \sqrt[3]{3}$ Rewrite in radical form.

b. $\sqrt[4]{9z^2}$

$\sqrt[4]{9z^2} = (9z^2)^{\frac{1}{4}}$ Rational exponents

$= (3^2 \cdot z^2)^{\frac{1}{4}}$ $9 = 3^2$

$= 3^{2\left(\frac{1}{4}\right)} \cdot z^{2\left(\frac{1}{4}\right)}$ Power of a Power

$= 3^{\frac{1}{2}} \cdot z^{\frac{1}{2}}$ Multiply.

$= \sqrt{3} \cdot \sqrt{z}$ $3^{\frac{1}{2}} = \sqrt{3}$, $z^{\frac{1}{2}} = \sqrt{z}$

$= \sqrt{3z}$ Simplify.

c. $\dfrac{m^{\frac{1}{2}} - 1}{m^{\frac{1}{2}} + 1}$

$\dfrac{m^{\frac{1}{2}} - 1}{m^{\frac{1}{2}} + 1} = \dfrac{m^{\frac{1}{2}} - 1}{m^{\frac{1}{2}} + 1} \cdot \dfrac{m^{\frac{1}{2}} - 1}{m^{\frac{1}{2}} - 1}$ $m^{\frac{1}{2}} - 1$ is the conjugate of $m^{\frac{1}{2}} + 1$.

$= \dfrac{m - 2m^{\frac{1}{2}} + 1}{m - 1}$ Multiply.

CHECK Your Progress

5A. $\dfrac{\sqrt[4]{32}}{\sqrt[3]{2}}$ **5B.** $\sqrt[3]{16x^4}$ **5C.** $\dfrac{y^{\frac{1}{2}} + 2}{y^{\frac{1}{2}} - 2}$

STANDARDIZED TEST EXAMPLE

6 If x is a positive number, then $\dfrac{x^{\frac{4}{3}} \cdot x^{\frac{2}{3}}}{x^{\frac{1}{3}}} = ?$

A $x^{\frac{2}{3}}$ B $x^{\frac{3}{2}}$ C $\sqrt[3]{x^5}$ D $\sqrt[5]{x^3}$

$\dfrac{x^{\frac{4}{3}} \cdot x^{\frac{2}{3}}}{x^{\frac{1}{3}}} = \dfrac{x^{\frac{6}{3}}}{x^{\frac{1}{3}}}$ Add the exponents in the numerator.

$= \dfrac{x^2}{x^{\frac{1}{3}}}$ Simplify.

$= x^2 \cdot x^{-\frac{1}{3}}$ Quotient of Powers

$= x^{\frac{5}{3}}$ or $\sqrt[3]{x^5}$ The answer is C.

Test-Taking Tip

When working with rational exponents, the rules are the same as when you add, subtract, or multiply fractions.

CHECK Your Progress

6. If y is positive, then $\frac{y \cdot y^2}{y^{\frac{1}{2}}} = ?$

F $y^{\frac{3}{2}}$ **G** $y^{\frac{5}{2}}$ **H** $\sqrt[2]{y^3}$ **J** $\sqrt[5]{y^2}$

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CONCEPT SUMMARY *Expressions with Rational Exponents*

An expression with rational exponents is simplified when all of the following conditions are met.

- It has no negative exponents.
- It has no fractional exponents in the denominator.
- It is not a complex fraction.
- The index of any remaining radical is the least number possible.

CHECK Your Understanding

Example 1 (p. 415)

Write each expression in radical form.

1. $7^{\frac{1}{3}}$ **2.** $x^{\frac{2}{3}}$

Write each radical using rational exponents.

3. $\sqrt[4]{26}$ **4.** $\sqrt[3]{6x^5y^7}$

Example 2 (p. 416)

Evaluate each expression.

5. $125^{\frac{1}{3}}$ **6.** $81^{-\frac{1}{4}}$ **7.** $27^{\frac{2}{3}}$ **8.** $\frac{54}{9^{\frac{3}{2}}}$

Example 3 (p. 417)

9. ECONOMICS When inflation causes the price of an item to increase, the new cost C and the original cost c are related by the formula $C = c(1 + r)^n$, where r is the rate of inflation per year as a decimal and n is the number of years. What would be the price of a \$4.99 item after six months of 5% inflation?

Examples 4, 5 (pp. 417–418)

Simplify each expression.

10. $a^{\frac{2}{3}} \cdot a^{\frac{1}{4}}$ **11.** $\frac{x^{\frac{5}{6}}}{x^{\frac{1}{6}}}$ **12.** $\frac{a^2}{b^{\frac{1}{3}}} \cdot \frac{b}{a^{\frac{1}{2}}}$

13. $\sqrt[6]{27x^3}$ **14.** $\frac{\sqrt[4]{27}}{\sqrt[4]{3}}$ **15.** $\frac{x^{\frac{1}{2}} + 1}{x^{\frac{1}{2}} - 1}$

Example 6 (pp. 418–419)

16. STANDARDIZED TEST PRACTICE If a is positive, then $\frac{a^5 \cdot a^{\frac{2}{3}}}{a^{\frac{4}{3}}} = ?$

A a **B** a^2 **C** $\sqrt[3]{a^{13}}$ **D** $\sqrt[13]{a^3}$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
17–24	1
25–32	2
33, 34	3
35–42, 50	4
43–49	5
51, 52	6

Write each expression in radical form.

17. $6^{\frac{1}{5}}$ **18.** $4^{\frac{1}{3}}$ **19.** $c^{\frac{2}{5}}$ **20.** $(x^2)^{\frac{4}{3}}$

Write each radical using rational exponents.

21. $\sqrt{23}$ **22.** $\sqrt[3]{62}$ **23.** $\sqrt[4]{16z^2}$ **24.** $\sqrt[3]{5x^2y}$

Evaluate each expression.

25. $16^{\frac{1}{4}}$ **26.** $216^{\frac{1}{3}}$ **27.** $25^{-\frac{1}{2}}$ **28.** $(-27)^{-\frac{2}{3}}$

29. $81^{-\frac{1}{2}} \cdot 81^{\frac{3}{2}}$ **30.** $8^{\frac{3}{2}} \cdot 8^{\frac{5}{2}}$ **31.** $\dfrac{16^{\frac{1}{2}}}{9^{\frac{1}{2}}}$ **32.** $\dfrac{8^{\frac{1}{3}}}{64^{\frac{1}{3}}}$

BASKETBALL For Exercises 33 and 34, use the following information.
A women's regulation-sized basketball is slightly smaller than a men's basketball. The radius r of the ball that holds V volume of air is $r = \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}}$.

33. Find the radius of a women's basketball if it will hold 413 cubic inches of air.

34. Find the radius of a men's basketball if it will hold 455 cubic inches of air.

Simplify each expression.

35. $y^{\frac{5}{3}} \cdot y^{\frac{7}{3}}$ **36.** $x^{\frac{3}{4}} \cdot x^{\frac{9}{4}}$ **37.** $\left(b^{\frac{1}{3}}\right)^{\frac{3}{5}}$ **38.** $\left(a^{-\frac{2}{3}}\right)^{-\frac{1}{6}}$

39. $w^{-\frac{4}{5}}$ **40.** $\dfrac{r^{\frac{2}{3}}}{r^{\frac{1}{6}}}$ **41.** $\dfrac{a^{-\frac{1}{2}}}{6a^{\frac{1}{3}} \cdot a^{-\frac{1}{4}}}$ **42.** $\dfrac{y^{\frac{3}{2}}}{y^{\frac{1}{2}} + 2}$

43. $\sqrt[4]{25}$ **44.** $\sqrt[6]{27}$ **45.** $\sqrt{17} \cdot \sqrt[3]{17^2}$ **46.** $\sqrt[8]{25x^4y^4}$

47. $\dfrac{xy}{\sqrt{z}}$ **48.** $\sqrt[3]{\sqrt{8}}$ **49.** $\dfrac{8^{\frac{1}{6}} - 9^{\frac{1}{4}}}{\sqrt{3} + \sqrt{2}}$ **50.** $\dfrac{x^{\frac{5}{3}} - x^{\frac{1}{3}}z^{\frac{4}{3}}}{x^{\frac{2}{3}} + z^{\frac{2}{3}}}$

51. GEOMETRY A triangle has a base of $3r^{\frac{1}{2}}s^{\frac{1}{4}}$ units and a height of $4r^{\frac{1}{4}}s^{\frac{1}{2}}$ units. Find the area of the triangle.

52. GEOMETRY Find the area of a circle whose radius is $3x^{\frac{2}{3}}y^{\frac{1}{5}}z^2$ centimeters.

53. Find the simplified form of $32^{\frac{1}{2}} + 3^{\frac{1}{2}} - 8^{\frac{1}{2}}$.

54. What is the simplified form of $81^{\frac{1}{3}} - 24^{\frac{1}{3}} + 3^{\frac{1}{3}}$?

55. BIOLOGY Suppose a culture has 100 bacteria to begin with and the number of bacteria doubles every 2 hours. Then the number N of bacteria after t hours is given by $N = 100 \cdot 2^{\frac{t}{2}}$. How many bacteria will be present after $3\frac{1}{2}$ hours?

EXTRA PRACTICE
See pages 907 and 932.
Math Online
Self-Check Quiz at
algebra2.com

H.O.T. Problems

56. OPEN ENDED Determine a value of b for which $b^{\frac{1}{6}}$ is an integer.

57. REASONING Explain why $(-16)^{\frac{1}{2}}$ is not a real number.

58. CHALLENGE Explain how to solve $9^x = 3^{x + \frac{1}{2}}$ for x.

59. REASONING Determine whether $\sqrt[n]{b^m} = \left(\sqrt[n]{b}\right)^m$ is *always*, *sometimes*, or *never* true. Explain.

60. *Writing in Math* Refer to the information on page 415 to explain how rational exponents can be applied to astronomy. Explain how to write the formula $r = D\left(\frac{M_p}{M_S}\right)^{\frac{2}{5}}$ in radical form and simplify it.

STANDARDIZED TEST PRACTICE

61. ACT/SAT If $3^5 \cdot p = 3^3$, then $p =$

A -3^2

B 3^{-2}

C $\frac{1}{3}$

D $3^{\frac{1}{3}}$

62. REVIEW Which of the following sentences is true about the graphs of $y = 2(x - 3)^2 + 1$ and $y = 2(x + 3)^2 + 1$?

F Their vertices are maximums

G The graphs have the same shape with different vertices.

H The graphs have different shapes with different vertices.

J One graph has a vertex that is a maximum while the other graph has a vertex that is a minimum.

Spiral Review

Simplify. (Lessons 7-4 and 7-5)

63. $\sqrt{4x^3y^2}$

64. $\left(2\sqrt{6}\right)\left(3\sqrt{12}\right)$

65. $\sqrt{32} + \sqrt{18} - \sqrt{50}$

66. $\sqrt[4]{(-8)^4}$

67. $\sqrt[4]{(x-5)^2}$

68. $\sqrt{\frac{9}{36}x^4}$

TEMPERATURE For Exercises 69 and 70, use the following information.
There are three temperature scales: Fahrenheit (°F), Celsius (°C), and Kelvin (K). The function $K(C) = C + 273$ can be used to convert Celsius temperatures to Kelvin. The function $C(F) = \frac{5}{9}(F - 32)$ can be used to convert Fahrenheit temperatures to Celsius. (Lesson 7-1)

69. Write a composition of functions that could be used to convert Fahrenheit temperatures to Kelvin.

70. Find the temperature in Kelvin for the boiling point of water and the freezing point of water if water boils at 212°F and freezes at 32°F.

71. PHYSICS A toy rocket is fired upward from the top of a 200-foot tower at a velocity of 80 feet per second. The height of the rocket t seconds after firing is given by the formula $h(t) = -16t^2 + 80t + 200$. Find the time at which the rocket reaches its maximum height of 300 feet. (Lesson 5-6)

GET READY for the Next Lesson

PREREQUISITE SKILL Find each power. (Lesson 7-5)

72. $\left(\sqrt{x-2}\right)^2$

73. $\left(\sqrt[3]{2x-3}\right)^3$

74. $\left(\sqrt{x}+1\right)^2$

75. $\left(2\sqrt{x}-3\right)^2$